

Osculating varieties of Veronesean and their higher secant varieties.

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Abstract. We consider the varieties $O_{k,n,d}$ of the k -osculating spaces to the Veronese varieties, the d -uple embeddings of \mathbb{P}^n ; we study the dimension of their higher secant varieties. Via inverse systems (apolarity) and the study of certain spaces of forms we are able, in several cases, to determine whether those secant varieties are defective or not.

0. Introduction.

Let us consider the following case of a quite classical problem: given a generic form f of degree d in $R := k[x_0, \dots, x_n]$, what is the minimum s for which it is possible to write $f = L_1^{d-k} F_1 + \dots + L_s^{d-k} F_s$, where $L_i \in R_1$ and $F_i \in R_k$? When $k = 0$ this is known as “Waring problem for forms” (the original Waring problem is for integers), and it has been solved via results in [AH], e.g. see [IK] or [Ge].

In its generality, this is what was classically called “to find canonical forms for a $(n+1)$ -ary d -ic” (e.g. see [W]).

We will study this problem here via the study of the dimension of higher secant varieties to osculating varieties of Veronesean, since this geometrical problem is equivalent to the one stated before.

1. Preliminaries.

1.1. Notation.

i) In the following we set $R := k[x_0, \dots, x_n]$, where $k = \bar{k}$ and $\text{char} k = 0$, hence R_d will denote the forms of degree d on \mathbb{P}^n .

ii) If $X \subseteq \mathbb{P}^N$ is an irreducible projective variety, an m -fat point on X is the $(m-1)^{th}$ infinitesimal neighborhood of a smooth point P in X , and it will be denoted by mP (i.e. the scheme mP is defined by the ideal sheaf $\mathcal{I}_{P,X}^m \subset \mathcal{O}_X$).

Let $\dim X = n$; then, mP is a 0-dimensional scheme of length $\binom{m-1+n}{n}$.

If Z is the union of the $(m-1)^{th}$ -infinitesimal neighborhoods in X of s generic points of X , we shall say for short that Z is union of s generic m -fat points on X .

iii) If $X \subseteq \mathbb{P}^N$ is a variety and P is a smooth point on it, the projectivized tangent space to X at P is denoted by $T_{X,P}$.

iv) We denote by $\langle U, V \rangle$ both the linear span in a vector space or in a projective space of two linear subspaces U, V .

v) If X is a 0-dimensional scheme, we denote by $l(X)$ its length, while its support is denoted by $\text{supp} X$.

1.2. Definition. Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)^{th}$ *higher secant variety* of X is the closure of the union of all linear spaces spanned by s points of X , and it will be denoted by X^s . Let $\dim X = n$; the *expected dimension* for X^s is

$$\expdim X^s := \min\{N, sn + s - 1\}$$

where the number $sn + s - 1$ corresponds to ∞^{sn} choices of s points on X , plus ∞^{s-1} choices of a point on the \mathbb{P}^{s-1} spanned by the s points. When this number is too big, we expect that $X^s = \mathbb{P}^N$. Since it is not always the case that X^s has the expected dimension, when $\dim X^s < \min\{N, sn + s - 1\}$, X^s is said to be *defective*.

A classical result about secant varieties is Terracini's Lemma (see [Te], or, e.g. [A]), which we give here in the following form:

1.3. Terracini's Lemma: *Let X be an irreducible variety in \mathbb{P}^N , and let P_1, \dots, P_s be s generic points on X . Then, the projectivised tangent space to X^s at a generic point $Q \in \langle P_1, \dots, P_s \rangle$ is the linear span in \mathbb{P}^N of the tangent spaces T_{X, P_i} to X at P_i , $i = 1, \dots, s$, hence*

$$\dim X^s = \dim \langle T_{X, P_1}, \dots, T_{X, P_s} \rangle.$$

1.4. Corollary. *Let (X, \mathcal{L}) be an integral, polarized scheme. If \mathcal{L} embeds X as a closed scheme in \mathbb{P}^N , then*

$$\dim X^s = N - \dim h^0(\mathcal{I}_{Z, X} \otimes \mathcal{L})$$

where Z is union of s generic 2-fat points in X .

Proof. By Terracini's Lemma, $\dim X^s = \dim \langle T_{X, P_1}, \dots, T_{X, P_s} \rangle$, with P_1, \dots, P_s generic points on X . Since X is embedded in $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})^*)$, we can view the elements of $H^0(X, \mathcal{L})$ as hyperplanes in \mathbb{P}^N ; the hyperplanes which contain a space T_{X, P_i} correspond to elements in $H^0(\mathcal{I}_{2P_i, X} \otimes \mathcal{L})$, since they intersect X in a subscheme containing the first infinitesimal neighborhood of P_i . Hence the hyperplanes of \mathbb{P}^N containing the subspace $\langle T_{X, P_1}, \dots, T_{X, P_s} \rangle$ are the sections of $H^0(\mathcal{I}_{Z, X} \otimes \mathcal{L})$, where Z is the scheme union of the first infinitesimal neighborhoods in X of the points P_i 's. \square

1.5. Definition. Let $X \subset \mathbb{P}^N$ be a variety, and let $P \in X$ be a smooth point; we define the k^{th} *osculating space to X at P* as the linear space generated by $(k+1)P$, and we denote it by $O_{k, X, P}$; hence $O_{0, X, P} = \{P\}$, and $O_{1, X, P} = T_{X, P}$, the projectivised tangent space to X at P .

Let $X_0 \subset X$ be the dense set of the smooth points where $O_{k, X, P}$ has maximal dimension. The k^{th} *osculating variety to X* is defined as:

$$O_{k, X} = \overline{\bigcup_{P \in X_0} O_{k, X, P}}.$$

2. Osculating varieties to Veronesean, and their higher secant varieties.

2.1. Notation.

i) We will consider here Veronese varieties, i.e. embeddings of \mathbb{P}^n defined by the linear system of all forms of a given degree d : $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$. The d -ple Veronese embedding of \mathbb{P}^n , i.e. $\text{Im} \nu_d$, will be denoted by $X_{n,d}$.

ii) In the following we set $O_{k,n,d} := O_{k,X_{n,d}}$, so that the $(s-1)^{\text{th}}$ higher secant variety to the k^{th} osculating variety to the Veronese variety $X_{n,d}$ will be denoted by $O_{k,n,d}^s$.

2.2. Remark. From now on $\mathbb{P}^N = \mathbb{P}(R_d)$; a form M will denote, depending on the situation, a vector in R_d or a point in \mathbb{P}^N .

We can view $X_{n,d}$ as given by the map $(\mathbb{P}^n)^* \rightarrow \mathbb{P}^N$, where $L \rightarrow L^d$, $L \in R_1$. Hence

$$X_{n,d} = \{L^d, \quad L \in R_1\}.$$

Let us assume (and from now on this assumption will be implicit) that $d \geq k$; at the point $P = L^d$ we have (see [Se], [CGG] sec.1, [BF] sec.2):

$$O_{k,X_{n,d},P} = \{L^{d-k}F, \quad F \in R_k\}. \quad (*)$$

Notice that $O_{k,X_{n,d},P}$ has maximal dimension $\dim R_k - 1 = \binom{k+n}{n} - 1$ for all $P \in X_{n,d}$. This can also be seen in the following way: the fat point $(k+1)P$ on $X_{n,d}$ gives independent conditions to the hyperplanes of \mathbb{P}^N , since it gives independent conditions to the forms of degree d in \mathbb{P}^n .

Hence, $O_{k,n,d} = \bigcup_{P \in X_{n,d}} O_{k,X_{n,d},P}$.

As we have already noticed, for $k = 0$ $(*)$ gives $O_{k,X_{n,d},P} = \{P\} = \{L^d\}$, and for $k = 1$ it becomes $O_{k,X_{n,d},P} = T_{X_{n,d},P} = \{L^{d-1}F, \quad F \in R_1\}$.

In general, we have:

$$O_{k,n,d} = \{L^{d-k}F, \quad L \in R_1, \quad F \in R_k\}.$$

Hence,

$$O_{k,n,d}^s = \{L_1^{d-k}F_1 + \dots + L_s^{d-k}F_s, \quad L_i \in R_1, \quad F_i \in R_k, \quad i = 1, \dots, s\}.$$

In the following we also need to know the tangent space $T_{O_{k,n,d},Q}$ of $O_{k,n,d}$ at the generic point $Q = L^{d-k}F$ (with $L \in R_1$, $F \in R_k$); one has that the affine cone over $T_{O_{k,n,d},Q}$ is $W = W(L, F) = \langle L^{d-k}R_k, L^{d-k-1}FR_1 \rangle$ (see [CGG] sec.1, [BF] sec.2).

2.3. Lemma. The dimension of $O_{k,n,d}$ is always the expected one, that is,

$$\dim O_{k,n,d} = \min\{N, n + \binom{k+n}{n} - 1\}$$

Proof. By 2.2, $\dim O_{k,n,d} = \dim W(L, F) - 1$, for a generic choice of L, F , so that we can assume that L does not divide F . When $\mathbb{P}(W) \neq \mathbb{P}^N$, we have $\dim W = \dim L^{d-k}R_k + \dim L^{d-k-1}FR_1 - \dim L^{d-k}R_k \cap$

$L^{d-k-1}FR_1 = \binom{k+n}{n} + (n+1) - 1 = \binom{k+n}{n} + n$, since there is only the obvious relation between LR_k and FR_1 , namely $LF - FL = 0$.

2.4. Consider the classic Waring problem for forms, i.e. “if we want to write a form of degree d as a sum of powers of linear forms, how many of them are necessary?” The problem is completely solved. In fact, $X_{n,d}^s = \{L_1^d + \dots + L_s^d, \quad L_i \in R_1\}$ (see previous remark), hence the Waring problem is equivalent to the problem of computing $\dim X_{n,d}^s$.

By Corollary 1.4 we have that $\dim X_{n,d}^s = N - \dim H^0(\mathcal{I}_{Z,\mathbb{P}^n} \otimes \mathcal{O}(d)) = H(Z, d) - 1$, where Z is a scheme of s generic 2-fat points in \mathbb{P}^n , and $H(Z, d)$ is the Hilbert function of Z in degree d . Since $H(Z, d)$ is completely known (see [AH]), we are done.

More generally, one could ask which is the least s such that a form of degree d can be written as $L_1^{d-k}F_1 + \dots + L_s^{d-k}F_s$, with $L_i \in R_1$ and $F_i \in R_k$ for $i = 1, \dots, s$; since by Remark 2.2 the variety $O_{k,n,d}^s$ parameterizes exactly the forms in R_d which can be written in this way, this is equivalent to answering, for each k, n, d , to the following question:

Find the least s , for each k, n, d , for which $O_{k,n,d}^s = \mathbb{P}^N$.

We are interested in a more complete description of the stratification of the forms of degree d parameterized by those varieties, namely in answering the following question:

Describe all s for which $O_{k,n,d}^s$ is defective, i.e. for which $\dim O_{k,n,d}^s < \exp \dim O_{k,n,d}^s$.

Notice that, since $d \geq k$, one has $\dim O_{k,n,d} = N$ if and only if $\binom{d+n}{n} \leq n + \binom{k+n}{n}$, hence for all such k, n, d and for any s we have $\dim O_{k,n,d}^s = \exp \dim O_{k,n,d}^s = N$.

So we have to study this problem when $\binom{d+n}{n} > n + \binom{k+n}{n}$, $s \geq 2$; it is easy to check that whenever $n \geq 2$ this condition is equivalent to $d \geq k + 1$; on the other hand the case $n = 1$ (osculating varieties of rational normal curves) can be easily described (all the $O_{k,1,d}^s$'s have the expected dimension, see next section), thus the question becomes:

Question Q(k,n,d): *For all k, n, d such that $d \geq k + 1$, $n \geq 2$, describe all s for which*

$$\dim O_{k,n,d}^s < \min\{N, s(n + \binom{k+n}{n} - 1) + s - 1\} = \min\left\{\binom{d+n}{n} - 1, s\binom{k+n}{n} + sn - 1\right\}.$$

2.5. Remark. Terracini's Lemma 1.4 says that $\dim O_{k,n,d}^s = N - h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(1))$, where X is a generic union of 2-fat points on $O_{k,n,d}$; we are not able to handle directly the study of $h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(1))$, nevertheless, Terracini's Lemma 1.3 says that the tangent space of $O_{k,n,d}^s$ at a generic point of $\langle P_1, \dots, P_s \rangle$, $P_i \in O_{k,n,d}$, is the span of the tangent spaces of $O_{k,n,d}$ at P_i ; if $T_{O_{k,n,d}, P_i} = \mathbb{P}(W_i)$, then

$$\dim O_{k,n,d}^s = \dim \langle T_{O_{k,n,d}, P_1}, \dots, T_{O_{k,n,d}, P_s} \rangle = \dim \langle W_1, \dots, W_s \rangle - 1$$

We want to prove, via Macaulay's theory of “inverse systems”, (see [I], [IK], [Ge], [CGG], [BF]) that, for a single W_i , $\dim W_i = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Z(d))$ where $Z = Z(k, n)$ is a certain 0-dimensional scheme that we will

analyze further, and $\dim < W_1, \dots, W_s > = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Y(d))$ where $Y = Y(k, n, s)$ is a generic union in \mathbb{P}^n of s 0-dimensional schemes isomorphic to Z . Hence,

$$\dim O_{k,n,d}^s = \dim < W_1, \dots, W_s > - 1 = N - h^0(\mathbb{P}^n, \mathcal{I}_Y(d)).$$

So, one strategy in order to answer to the question $Q(k, n, d)$ for a given (k, n, d) is the following:

1st step: try to compute directly $\dim < W_1, \dots, W_s >$; if this is not possible, then

2nd step: use the theory of inverse systems (classically *apolarity*):

Compute $W^\perp \subset R_d$, with respect to the perfect pairing $\phi : R_d \times R_d \rightarrow k$, where:

- W is a vector subspace of R_d ,

- $\phi(f, g) := \sum_{I \in A_{n,d}} f_I g_I$, where $A_{n,d} := \{(i_0, \dots, i_n) \in \mathbb{N}^{n+1}, \sum_j i_j = d\}$, with any fixed ordering; this gives a monomial basis $\{x_0^{i_0} \cdot \dots \cdot x_n^{i_n}\}$ for the vector space R_d ; if $f \in R_d$, $f = \sum_{i_0, \dots, i_n \in A_{n,d}} f_{i_0, \dots, i_n} x_0^{i_0} \cdot \dots \cdot x_n^{i_n}$, we write for short $f = \sum f_I \mathbf{x}^I$, with $I = (i_0, \dots, i_n)$.

Then, consider $I_d := W^\perp \subset R_d$. It generates an ideal $(I_d) \subset R$; in this way we define the scheme $Z(k, n, d) \subset \mathbb{P}^n$ by setting: $I_{Z(k,n,d)} := (I_d)^{\text{sat}}$. We will show that these schemes do not depend on d .

3rd step, compute the postulation for a generic union of s schemes $Z(k, n, d)$ in \mathbb{P}^n .

Recall that $[< W_1, \dots, W_s >]^\perp = W_1^\perp \cap \dots \cap W_s^\perp$.

2.6. Lemma. *For all k, n and $d \geq k + 2$, we have:*

$$(k+1)O \subset Z(k, n, d) \subset (k+2)O,$$

where $Z(k, n, d)$ was defined in 2.5, and $O = \text{supp } Z(k, n, d) \in \mathbb{P}^n$.

Proof. Let $W = < L^{d-k}R_k, L^{d-k-1}FR_1 > \subset R_d$ be the affine cone over $T_{O_{k,n,d},Q}$ at a generic point $Q = L^{d-k}F$, with $L \in R_1$, $F \in R_k$. Without loss of generality we can choose $L = x_0$, so that $W = x_0^{d-k-1}(x_0R_k + FR_1)$, hence $x_0^{d-k}R_k \subset W \subset x_0^{d-k-1}R_{k+1}$. So for any (k, n, d) ,

$$(x_0^{d-k-1}R_{k+1})^\perp \subset W^\perp \subset (x_0^{d-k}R_k)^\perp. \quad (**)$$

Now, denoting by \mathfrak{p} the ideal (x_1, \dots, x_n) , we have:

$$\begin{aligned} (x_0^{d-t}R_t)^\perp &= < \{x_0^{i_0} \cdot \dots \cdot x_n^{i_n} \mid \sum_j i_j = d, i_0 \leq d-t-1\} > = \\ &< (\mathfrak{p}^d)_d, x_0(\mathfrak{p}^{d-1})_{d-1}, \dots, x_0^{d-t-1}(\mathfrak{p}^{t+1})_{t+1} > = (\mathfrak{p}^{t+1})_d. \end{aligned}$$

Now let us view everything in $(**)$ as the degree d part of a homogeneous ideal; we get:

$$(\mathfrak{p}^{k+2})_d \subset (I_{Z(k,n,d)})_d \subset (\mathfrak{p}^{k+1})_d.$$

Let (x_1, \dots, x_n) be local coordinates in \mathbb{P}^n around the point $O = (1, 0, \dots, 0)$; the above inclusions give, in terms of 0-dimensional schemes in \mathbb{P}^n :

$$(k+1)O \subset Z(k, n, d) \subset (k+2)O.$$

2.7. Lemma. For any k, n, d with $d \geq k + 2$ the length of $Z = Z(k, n, d)$ is:

$$l(Z) = \dim W = \binom{k+n}{n} + n.$$

Proof. We have seen that $Z(k, n, d) \subset (k+2)O$, with O a point in \mathbb{P}^n (notice that this part of the inclusions in 2.6 works also for $d = k + 1$); setting $X := (k+2)O$, $d \geq k + 1$ then gives $\binom{d+n}{n} \geq l(X) = \binom{k+1+n}{n} \geq l(Z)$. We have ($W \neq R_d$ by assumption) $\dim I_d = \dim W^\perp = \binom{d+n}{n} - \dim W$, hence if we prove that $\dim I_d = \binom{d+n}{n} - l(Z)$, that is, if Z imposes independent conditions to the forms of degree d , the thesis will follow. One $(k+2)$ -fat point always imposes independent conditions to the forms of degree $d \geq k + 1$. Since $Z \subset X = (k+2)O$, then $h^1(\mathcal{I}_Z(d)) = 0$ immediately follows.

Now we have seen that our problem can be translated into a problem of studying certain schemes $Z(k, n, d) \subset \mathbb{P}^n$; we want to check that actually these schemes are the same for all $d \geq k + 2$, say $Z(k, n, d) = Z(k, n)$.

2.8. Lemma. For any k, n and $d \geq k + 2$, we have $Z(k, n, d) = Z(k, n, k + 2)$. Henceforth we will denote $Z(k, n) = Z(k, n, d)$, for all $d \geq k + 2$.

Proof. By the previous lemmata we already know that $Z(k, n, d)$ and $Z(k, n, k + 2)$ have the same support and the same length, hence it is enough to show that $Z(k, n, d) \subset Z(k, n, k + 2)$ (as schemes) in order to conclude. This will be done if we check that $I(Z(k, n, k + 2))_d \subset I(Z(k, n, d))_d$; in fact, since both ideals are generated in degrees $\leq d$, this will imply that $I(Z(k, n, k + 2))_j \subset I(Z(k, n, d))_j$, $\forall j \geq d$, hence the inclusion will hold also between the two saturations, implying $Z(k, n, d) \subset Z(k, n, k + 2)$.

Let $f \in I(Z(k, n, k + 2))_d$, then $f = h_1 g_1 + \dots + h_r g_r$, where $h_j \in R_{d-k-2}$ and $g_j \in I(Z(k, n, k + 2))_{k+2}$; since $I(Z(k, n, d))_d$ is the perpendicular to $W = \langle L^{d-k} R_k, L^{d-k-1} F R_1 \rangle$, it is enough to check that $h_j g_j \in W^\perp$, $j = 1, \dots, r$. Without loss of generality we can assume $L = x_0$; hence, since $g_j \in \langle L^2 R_k, L F R_1 \rangle^\perp$, $g_j = x_0 g' + g''$, with $g', g'' \in k[x_1, \dots, x_n]$ and $g' \in (F R_1)^\perp$. It will be enough to prove $x_0^{i_0} \dots x_n^{i_n} g_j = x_0^{i_0+1} \dots x_n^{i_n} g' + x_0^{i_0} \dots x_n^{i_n} g'' \in W^\perp$, $\forall i_0, \dots, i_n$ such that $i_0 + \dots + i_n = d - k - 2$. It is clear that $x_0^{i_0} \dots x_n^{i_n} g'' \in W^\perp$, since $i_0 \leq d - k - 2$; on the other hand, $x_0^{i_0+1} \dots x_n^{i_n} g' \in (x_0^{d-k} R_k)^\perp$ again by looking at the degree of x_0 , while $x_0^{i_0+1} \dots x_n^{i_n} g' \in (x_0^{d-k-1} F R_1)^\perp$ since $g' \in (F R_1)^\perp$.

2.9. Remark. From the lemmata above it follows that in order to study the dimension of $\mathcal{O}_{k,n,d}^s$, $\forall d \geq k + 2$, we only need to study the postulation of unions of schemes $Z(k, n)$. For $d = k + 1$, we will work directly on W , see Proposition 3.4.

What we got is a sort of “generalized Terracini” for osculating varieties to Veronesean, since the formula $\dim \mathcal{O}_{k,n,d}^s = N - h^0(\mathcal{I}_Y(d))$ reduces to the one in Corollary 1.4 for $k = 0$. Instead of studying 2-fat points on $X_{n,d}^s$ (see Remark 2.5), we can study the schemes $Y \subset \mathbb{P}^n$.

2.10. Notation. Let $Y \subset \mathbb{P}^n$ be a 0-dimensional scheme; we say that Y is *regular* in degree d , $d \geq 0$, if the restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(\mathcal{O}_Y(d))$ has maximal rank, i.e. if $h^0(\mathcal{I}_Y(d)) \cdot h^1(\mathcal{I}_Y(d)) = 0$. We set $\exp h^0(\mathcal{I}_Y(d)) := \max \{0, \binom{d+n}{n} - l(Y)\}$; hence to say that Y is regular in degree d amounts to saying that $h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_Y(d))$.

Since we always have $h^0(\mathcal{I}_Y(d)) \geq \exp h^0(\mathcal{I}_Y(d))$, we write

$$h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_Y(d)) + \delta,$$

where $\delta = \delta(Y, d)$; hence whenever $\binom{d+n}{n} - l(Y) \geq 0$, we have $\delta = h^1(\mathcal{I}_Y(d))$, while if $\binom{d+n}{n} - l(Y) \leq 0$, $\delta = \binom{d+n}{n} - l(Y) + h^1(\mathcal{I}_Y(d))$; in any case, by setting $\exp h^1(\mathcal{I}_Y(d)) := \max \{0, l(Y) - \binom{d+n}{n}\}$, we get: $h^1(\mathcal{I}_Y(d)) = \exp h^1(\mathcal{I}_Y(d)) + \delta$.

2.11. Remark. For any k, n, d such that $d \geq k + 1$, let $Y = Y(k, n, s) \subset \mathbb{P}^n$ be the 0-dimensional scheme defined in 2.5 for $Z = Z(k, n)$, and $\delta = \delta(Y, d)$. Then

$$\dim O_{k,n,d}^s = \exp \dim O_{k,n,d}^s - \delta.$$

In particular, $\dim O_{k,n,d}^s = \exp \dim O_{k,n,d}^s$ if and only if:

$$h^0(\mathcal{I}_Y(d)) = 0, \quad \text{when } \binom{d+n}{n} \leq s \binom{k+n}{n} + sn;$$

$$h^0(\mathcal{I}_Y(d)) = N + 1 - l(Y) = \binom{d+n}{n} - s \binom{k+n}{n} - sn \quad (\text{i.e. } h^1(\mathcal{I}_Y(d)) = 0), \quad \text{when } \binom{d+n}{n} \geq s \binom{k+n}{n} + sn.$$

In fact $h^0(\mathcal{I}_Y(d)) = \ker \rho$ and $l(Y) = s \binom{k+n}{n} + sn$ (lemma 2.7), $\exp \dim O_{k,n,d}^s = \min \{N = \binom{d+n}{n} - 1, s(n + \binom{k+n}{n}) - 1\}$, and $\dim O_{k,n,d}^s = N - h^0(\mathcal{I}_Y(d)) = N - \exp h^0(\mathcal{I}_Y(d)) - \delta$ (see 2.5).

3. A few results and a conjecture.

Let us consider first the cases where the question $\mathbf{Q}(\mathbf{k}, \mathbf{n}, \mathbf{d})$ has been answered.

$\mathbf{Q}(\mathbf{k}, \mathbf{1}, \mathbf{d})$. In this case every $O_{k,1,d}^s$, with $d \geq k + 2$, has the expected dimension; in fact here $Z(k, 1) = (k+2)O$, and the scheme $Y = \{s(k+2)\text{-fat points}\} \in \mathbb{P}^1$ is regular in any degree d . Notice that for $d = k + 1$ we trivially have $O_{k,1,k+1} = \mathbb{P}^N$.

$\mathbf{Q}(\mathbf{1}, \mathbf{n}, \mathbf{d})$. Here the variety $O_{1,n,d}$ is the tangential variety to the Veronese $X_{n,d}$. It is shown in [CGG] that $Z(1, n)$ is a “(2, 3)–scheme” (i.e. the intersection in \mathbb{P}^n of a 3-fat point with a double line); this is easy to see, e.g. by choosing coordinates so that $L = x_0$, $F = x_1$.

The postulation of generic unions of such schemes in \mathbb{P}^n , and hence the defectivity of $O_{1,n,d}^s$, has been studied. Moreover, a conjecture regarding all defective cases is stated there:

Conjecture ([CGG]). $O_{1,n,d}^s$ is not defective, except in the following cases:

- 1) for $d = 2$ and $n \geq 2s$;
- 2) for $d = 3$ and $n = s = 2, 3, 4$.

In [CGG] the conjecture is proved for $s \leq 5$ (any d, n), for $d = 2$ (any s, n), for $d \geq 3$ and $n \geq s + 1$, for $d \geq 4$ and $s = n$, for $s \geq \frac{1}{3} \binom{n+2}{2} + 1$. In [B], the conjecture is proved for $n = 2, 3$ (any s, d).

$\mathbf{Q}(\mathbf{2}, \mathbf{2}, \mathbf{d})$. In [BF] it is proved that, for any $(s, d) \neq (2, 4)$, $O_{2,2,d}^s$ has the expected dimension.

Now we are going to prove some other cases.

The following (quite immediate) lemma describes what can be deduced about the postulation of the scheme Y from information on fat points:

3.1 Lemma. *Let P_1, \dots, P_s be generic points in \mathbb{P}^n , and set $X := (k+1)P_1 \cup \dots \cup (k+1)P_s$, $T := (k+2)P_1 \cup \dots \cup (k+2)P_s$. Now let Z_i be a 0-dimensional scheme supported on P_i , $(k+1)P_i \subset Z_i \subset (k+2)P_i$, with $l(Z_i) = l((k+1)P_i) + n$ for each $i = 1, \dots, s$, and set $Y := Z_1 \cup \dots \cup Z_s$. Then:*

Y is regular in degree d if one of the following a) or b) holds:

- a) $\binom{d+n}{n} \geq s \binom{k+n+1}{n}$, and $h^1(\mathcal{I}_T(d)) = 0$;
- b) $\binom{d+n}{n} \leq s \binom{k+n}{n}$, and $h^0(\mathcal{I}_X(d)) = 0$.

Y is not regular in degree d , with defectivity δ , if one of the following c) or d) holds:

- c) $h^1(\mathcal{I}_X(d)) > \exp h^1(\mathcal{I}_Y(d))$; in this case $\delta \geq h^1(\mathcal{I}_X(d)) - \max\{0, l(Y) - \binom{d+n}{n}\}$.
- d) $h^0(\mathcal{I}_T(d)) > \exp h^0(\mathcal{I}_Y(d))$; in this case $\delta \geq h^0(\mathcal{I}_T(d)) - \max\{0, \binom{d+n}{n} - l(Y)\}$.

Proof. The statement follows by considering the cohomology of the exact sequences:

$$0 \rightarrow \mathcal{I}_T(d) \rightarrow \mathcal{I}_Y(d) \rightarrow \mathcal{I}_{Y,T}(d) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{I}_Y(d) \rightarrow \mathcal{I}_X(d) \rightarrow \mathcal{I}_{X,Y}(d) \rightarrow 0$$

where we have: $h^1(\mathcal{I}_{Y,T}(d)) = h^1(\mathcal{I}_{X,Y}(d)) = 0$ since those two sheaves are supported on a 0-dimensional scheme.

3.2. Lemma. *Let $s \geq n+1$ and $d < k+1 + j \frac{k+1}{n}$, with $j = 2$ when $s \geq n+2$, and $j = 1$ for $s = n+1$. Then $O_{k,n,d}^s$ is not defective and $O_{k,n,d}^s = \mathbb{P}^N$.*

Proof. Let $Y \subset \mathbb{P}^n$ be as in 2.5; we have to prove that $h^0(\mathcal{I}_Y(d)) = 0$ in our hypotheses.

Let P_1, \dots, P_s be the support of Y ; we can always choose a rational normal curve $C \subset \mathbb{P}^n$ containing $n+2$ of the P_i 's (or just all of them if $s = n+1$). For any hypersurface F given by a section of $\mathcal{I}_Y(d)$, we have that either $C \subset F$, or $\deg(C \cap F) = nd$; hence, if $nd \leq (k+1)(n+j)$, where $j = 2$ when $s \geq n+2$ and $j = 1$ for $s = n+1$, by Bezout we get $C \subset F$. But this is precisely what our hypothesis on d says, hence $C \subset F$, but since we can always find a rational normal curve containing $n+3$ points in \mathbb{P}^n , this would imply that any $P \in \mathbb{P}^n$ is on F , i.e. $\mathcal{I}_Y(d) = 0$.

Q(k, 2, k+2). The following corollary describes this case completely:

3.3. Corollary. *Assume $d = k+2$ and $n = 2$. Then, $O_{k,n,d}^s$ is not defective for $s \geq 3$ and $k \geq 1$, and $O_{k,n,d}^s$ is defective for $s = 2$ and $k \geq 1$.*

Proof. By 3.2, $O_{k,2,k+2}^s$ is not defective for $s \geq 3$ and $d \geq 3$, i.e. $k \geq 2$; the case $k = 1$ is already known by [B].

For $s = 2$ and $k \geq 1$, let $Y = Y(k, 2) \subset \mathbb{P}^2$ be the 0-dimensional scheme defined in 2.5; it is easy to check that $\exp h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_T(d)) = 0$, T denoting the generic union of two $(k+2)$ -fat points in \mathbb{P}^2 . Since

T is not regular in degree $d = k + 2$ for any $k \geq 1$, we conclude by lemma 3.1 d) that $O_{k,n,k+2}^s$ is defective with defectivity $\geq h^0(\mathcal{I}_T(d)) = 1$ (the only section is given by the $k + 2$ -ple line through the two points).

The following results follow from direct computations on W .

$\mathbf{Q}(\mathbf{k}, \mathbf{n}, \mathbf{k} + 1)$. The following proposition describes this case completely.

3.4. Proposition. *If $s \geq 2$ and $d = k + 1$ then*

A) *if $s \leq n - 1$ and the expected dimension is $s \binom{k+n}{n} + sn$, then $O_{d-1,n,d}^s$ is defective with defect $\delta = s^2 - s + \sum_{h=2}^s (-1)^h \binom{s}{h} \binom{k-(h-1)+n}{n}$;*

B) *if $s \leq n - 1$ and the expected dimension is $\binom{d+n}{n}$ then*

i) *$O_{d-1,n,d}^s$ is defective with defect $\delta = \binom{n-s+d}{d} - s(n-s+1)$ if $s < \frac{1}{d} \binom{n-s+d}{d-1}$;*

ii) *$O_{d-1,n,d}^s = \mathbb{P}^N$ (i.e. $O_{d-1,n,d}^s$ is regular) if $s \geq \frac{1}{d} \binom{n-s+d}{d-1}$;*

C) *if $s \geq n$ then $O_{d-1,n,d}^s = \mathbb{P}^N$.*

Proof.

A) The variety $O_{d-1,n,d}^s$ wouldn't be defective if the only relations in $W_1 + \dots + W_s$ would be those we will be able to find in the proof of Proposition 3.5; what happens here is that there are too many relations: there are two kinds of them:

1) $x_i F_j \in \langle x_i R_k \rangle \cap \langle F_j R_1 \rangle$ for all $i = 0, \dots, s-1$ and $j = 1, \dots, s$. These relations are exactly s^2 ; then from these we get a defect of $s^2 - s$ (because the number of the allowed relations in order not to get defectivity is s);

2) $x_i x_j F \in \langle x_i R_k \rangle \cap \langle x_j R_k \rangle$ where $i \neq j \in \{0, \dots, s-1\}$ and $F \in R_{k-1}$. The defectivity δ of $O_{d-1,n,d}^s$ will be $\delta = s^2 - s + t$ where $t = \sum_{h=2}^s (-1)^h \binom{s}{h} \binom{k-(h-1)+n}{n}$ is the number of independent forms of type $x_i x_j F$ with $F \in R_{k-1}$. We can observe that t would be equal to $\binom{s}{2} \binom{k-1+n}{n}$ if for every F belonging to a base of R_{k-1} the forms $x_i x_j F$ were independent for all $i \neq j \in \{0, \dots, s-1\}$; but if $s > 2$ this is false: consider for example the following three forms $F_i = x_i G$, $F_j = x_j G$, $F_l = x_l G$ where $G \in R_{k-2}$ then $x_i x_j F_l = x_i x_l F_j = x_j x_l F_i$. Now t would be equal to $\binom{s}{2} \binom{k-1+n}{n} - \binom{s}{3} \binom{k-2+n}{n}$ if for every G belonging to a base of R_{k-2} the forms of type $x_i x_j x_l G$ were independent for all $i, j, l \in \{0, \dots, s-1\}$ with $i \neq j$, $i \neq l$ and $j \neq l$; but, as before, we can check that if $s > 3$, then $t \leq \binom{s}{2} \binom{k-1+n}{n} - \binom{s}{3} \binom{k-2+n}{n} + \binom{s}{4} \binom{k-3+n}{n}$. Proceeding in this way we eventually get that $t = \sum_{h=2}^s (-1)^h \binom{s}{h} \binom{k-(h-1)+n}{n}$.

We can conclude that in this case the defect is $\delta = s^2 - s + \sum_{h=2}^s (-1)^h \binom{s}{h} \binom{k-(h-1)+n}{n}$.

B) If $s \binom{n+d-1}{n} + ns \geq \binom{n+d}{n}$ we expect that $O_{d-1,n,d}^s = \mathbb{P}^N$. We have that $W_1 + \dots + W_s = \langle x_0 R_k, \dots, x_{s-1} R_k; F_1 R_1, \dots, F_s R_1 \rangle$ in $K[x_0, \dots, x_n]_d$. We can suppose F_i 's generic for any $i = 1, \dots, s$ in $K[x_s, \dots, x_n]_d$. Then $O_{d-1,n,d}^s = \mathbb{P}^N$ if and only if $\langle F_1 R_1, \dots, F_s R_1 \rangle \supset K[x_s, \dots, x_n]_d := S_d$; we can actually just consider the vector space $\langle F_1 S_1, \dots, F_s S_1 \rangle$; since the F_i 's are generic, its dimension will be $\min \left\{ \binom{n-s+d}{d}, s(n-s+1) \right\}$ (e.g. see [MMR]); hence we get that

i) if $s(n-s+1) < \binom{n-s+d}{d}$, then $O_{d-1,n,d}^s$ is defective. This happens if and only if $s < \frac{1}{d} \binom{n-s+d}{d-1}$. Then the defect is $\delta = \binom{n-s+d}{d} - s(n-s+1)$.

ii) if $s(n-s+1) \geq \binom{n-s+d}{d}$, then $O_{d-1,n,d}^s = \mathbb{P}^N$ (for example this is always true for $d \geq n$);

C) It suffices to prove that $O_{d-1,n,d}^s = \mathbb{P}^N$ for $s = n$.

If $s = n$ and $d = k+1$, the subspace $W_1 + \dots + W_s$ can be written as $\langle x_0 R_k, F_1 R_1, \dots, x_{n-1} R_k, F_n R_1 \rangle$, which turns out to be equal to $\langle x_0 R_k, \dots, x_{n-1} R_k, x_n^{k+1} \rangle = R_{k+1}$ so $O_{d-1,n,d}^n = \mathbb{P}^N$.

For $s \leq n+1$, we have several partial results:

3.5. Proposition. *If $s \leq n+1$ and $d \geq 2k+1$ then $O_{k,n,d}^s$ is regular.*

Proof. We have to study the dimension of the vector space $W_1 + \dots + W_s = \langle L_1^{d-k} R_k, L_1^{d-k-1} F_1 R_1, \dots, L_s^{d-k} R_k, L_s^{d-k-1} F_s R_1 \rangle$, where L_1, \dots, L_s are generic in R_1 and F_1, \dots, F_s are generic in R_k . Since $s \leq n+1$, without loss of generality we may suppose $L_i = x_{i-1}$ for $i = 1, \dots, s$. Since $d \geq 2k+1$, for $\beta = d-k \geq 0$, the vector space $W_1 + \dots + W_s$ can be written as $\langle x_0^{k+\beta+1} R_k, x_0^{k+\beta} F_1 R_1, \dots, x_{s-1}^{k+\beta+1} R_k, x_{s-1}^{k+\beta} F_s R_1 \rangle$. If we show that for a particular choice of $F_1, \dots, F_s \in R_k$ the dimension of $W_1 + \dots + W_s = \text{expdim}(O_{k,n,d}^s) + 1$ we can conclude by semi-continuity that $O_{k,n,d}^s$ has the expected dimension. Let us consider the case $\tilde{F}_i = x_i x_{i+1} \tilde{F}_i$ for $i = 1, \dots, s-2$, $F_{s-1} = x_{s-1} x_0 \tilde{F}_{s-1}$ and $F_s = x_0 x_1 \tilde{F}_s$, where the \tilde{F}_j 's are generic forms in R_{k-2} , $j = 1, \dots, n+1$. Let $\langle x_i^{k+\beta+1} R_k \rangle =: A_i$ and $\langle x_i^{k+\beta} F_{i+1} R_1 \rangle =: A'_i$, $i = 0, \dots, s-1$; then we get $A'_i = \langle x_i^{k+\beta} x_{i+1} x_{i+2} \tilde{F}_{i+1} R_1 \rangle$, $i = 0, \dots, s-3$; $A'_{s-2} = \langle x_{s-2}^{k+\beta} x_{s-1} x_0 \tilde{F}_{s-1} R_1 \rangle$ and $A'_{s-1} = \langle x_{s-1}^{k+\beta} x_0 x_1 \tilde{F}_s R_1 \rangle$. We can easily notice that $A_i \cap A_j = \{0\} = A'_i \cap A'_j$ for $i \neq j$ and $A_i \cap A'_i = \langle x_i^{k+\beta+1} x_{i+1} x_{i+2} \tilde{F}_{i+1} \rangle$, $i = 0, \dots, s-3$ (analogously if $i = s-2, s-1$). We can conclude that $\dim(W_1 + \dots + W_s) = s \binom{k+n}{n} + s(n+1) - s$, which is exactly the expected dimension.

3.6. Proposition. *If $s \leq n$ and $k+2 \leq d \leq 2k$ then $O_{k,n,d}^s$ is defective with defect δ such that:*

- A) $\delta \geq \binom{n-s+d}{d}$ if the expected dimension is $\binom{d+n}{n}$;
- B) $\delta \geq \binom{s}{2} \binom{2k-d+n}{n}$ if the expected dimension is $s \binom{k+n}{n} + sn$.

Proof. Let $\beta := d-k$; we can rewrite the vector space $W_1 + \dots + W_s$ as follows: $\langle x_0^\beta R_k, x_0^{\beta-1} F_1 R_1, \dots, x_{s-1}^\beta R_k, x_{s-1}^{\beta-1} F_s R_1 \rangle$.

A) We can observe that $(K[x_s, \dots, x_n])_d \cap (W_1 + \dots + W_s) = \{0\}$, so if we expect that $O_{k,n,d}^s = \mathbb{P}^N$ we get a defect $\delta \geq \binom{n-s+d}{d}$.

B) Suppose now that $s \left[\binom{k+n}{n} + n \right] < \binom{d+n}{n}$. If $O_{k,n,d}^s$ would have the expected dimension we would not be able to find more relations among the W_i 's than $x_i^\beta F_{i+1} \in \langle x_i^\beta R_k \rangle \cap \langle x_i^{\beta-1} F_{i+1} R_1 \rangle$, for $i = 0, \dots, s-1$ (as it happens in Proposition 3.5). But it's easy to see that $x_i^\beta x_j^\beta F \in \langle x_i^\beta R_k \rangle \cap \langle x_j^\beta R_k \rangle$ with $i \neq j$ and $F \in R_{k-\beta}$. We have exactly $\binom{s}{2}$ such terms for any choice of $F \in R_{k-\beta}$. We can also suppose that the $F_i \in R_k$ that appear in $W_1 + \dots + W_s$ are different from $x_j^\beta F$ for any $F \in R_{k-\beta}$ and $j = 0, \dots, s-1$ because F_1, \dots, F_s are generic forms of R_k . Then we can be sure that the form $x_i^\beta x_j^\beta F$ belonging to $\langle x_i^\beta R_k \rangle \cap \langle x_j^\beta R_k \rangle$ isn't one of the $x_i^\beta F_{i+1}$ that belongs to $\langle x_i^\beta R_k \rangle \cap \langle x_i^{\beta-1} F_{i+1} R_1 \rangle$. Now $\dim(R_{k-\beta}) = \binom{k-\beta+n}{n}$ then we can find $\binom{s}{2} \binom{k-\beta+n}{n}$ independent forms that give defectivity. Then in the case $s \left[\binom{k+n}{n} + n \right] < \binom{d+n}{n}$ we have $\dim(O_{k,n,d}^s) \leq \text{expdim} - \binom{s}{2} \binom{k-\beta+n}{n} = \text{expdim} - \binom{s}{2} \binom{2k-d+n}{n}$.

3.7. Proposition. *If $s = n+1$, $k+2 \leq d \leq 2k$ and $\text{expdim}(O_{k,n,d}^{n+1}) = (n+1) \left(\binom{k+n}{n} + n \right)$ then $O_{k,n,d}^{n+1}$ is defective with defect $\delta \geq \binom{n+1}{2} \binom{2k-d+n}{n}$.*

Proof. The proof of this fact is the same as case B) of the previous proposition.

3.8. Proposition. *If $s = n+1$, $n \geq \frac{k+2}{d-k-2}$, $k+2 < d \leq 2k$ and $\text{expdim}(O_{k,n,d}^{n+1}) = N$ then $O_{k,n,d}^{n+1}$ is defective with defect $\delta \geq \binom{(n+1)(d-k-1)-(d+1)}{n}$.*

Proof. If $k+2 < d \leq 2k$, then $2 < \beta := d-k \leq k$ and we have to study the dimension of $W_1 + \dots + W_{n+1} = \langle x_0^\beta R_k, x_0^{\beta-1} F_1 R_1, \dots, x_n^\beta R_k, x_n^{\beta-1} F_{n+1} R_1 \rangle$. If we expect that $O_{k,n,d}^{n+1} = \mathbb{P}^N$, it suffices to find a form in R_d which does not belong to $W_1 + \dots + W_{n+1}$. The forms we are looking for are:

- A) $x_0^{\beta_0} \dots x_n^{\beta_n}$ with $\sum_{i=0}^n \beta_i = d$ and $0 \leq \beta_i \leq \beta - 2$ for all $i \in \{0, \dots, n\}$, and
- B) $x_0^{\beta_0} \dots x_n^{\beta_n}$ with $\sum_{i=0}^n \beta_i = d$, at least one $\beta_i = \beta - 1$ and each of the others $\beta_j \leq \beta - 2$.

We will count only how many terms we can find in case A) and then we will conclude that the defectivity will be greater or equal to this number.

A) This case is equivalent to find forms of type $x_0^{d-(\gamma_0+k+2)} \dots x_n^{d-(\gamma_n+k+2)}$ with $\sum_{i=0}^n \gamma_i = nd - (n+1)(k+2)$ and $\gamma_i \geq 0$ for all $i = 0, \dots, n$. Then these forms are exactly $\binom{n+(n+1)(d-k-2)-d}{n} = \binom{(n+1)(d-k-1)-(d+1)}{n}$. This will be possible only if $(n+1)(d-k-2) - d \geq 0$ and so if $n \geq \frac{k+2}{d-k-2}$.

All the results on defectivity lead us to formulate the following:

3.9 Conjecture. $O_{k,n,d}^s$ is defective only if Y is as in case c) or d) of Lemma 3.1.

The conjecture amounts to say that the defectivity of Y can only occur if defectivity of the fat points schemes X or T imposes it.

In a forthcoming paper we intend to explore more in depth the connections between the postulation of fat points and our schemes Y .

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